LESSON 15 - STUDY GUIDE

Abstract. In this lesson we will finally solve the problem of recovering a function on $\mathbb{T}$ from its Fourier coefficients, by first analyzing the difficulties involved in summing the Fourier series from the convergence of its partial sums, and then by devising more effective summability methods, which are related to convolutions with approximate identities.

1. **Fourier series: summability methods.**

Study material: We will complete the study and analysis of ideas based on the topics contained in section **2 - Summability in Norm and Homogeneous Banach Spaces** from chapter **I - Fourier Series on** $\mathbb{T}$, corresponding to pgs. 8–16 in the second edition [1] and pgs. 9–17 in the third edition [2] of Katznelson's book, that we started in the previous lesson.

We finished the previous lesson by making the first observations about the convergence of the Fourier series of function a $f \in L^1(\mathbb{T})$,

$$S[f] \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int}, \tag{1.1}$$

with the goal of reconstructing the function from the sequence of its Fourier coefficients. Of course, the traditional way that immediately comes to mind for summing the series is to consider the limit of the sequence of its partial sums which, if one relates to the original real form of the Fourier series, with sines and cosines, corresponds to the symmetric complex form

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}(n)e^{int}. \tag{1.2}$$

However, a more abstract point of view of looking at (1.1) would lead us to consider the Fourier series as the Lebesgue integral of the sequence of Fourier coefficients $\{\hat{f}(n)\}_{n\in\mathbb{Z}}$, over the measure space $\mathbb{Z}$ with the counting measure. Abusing the notation a little bit, we could imagine the Fourier series as being the same as the integral

$$\int_{\mathbb{Z}} \hat{f}(n)e^{int}dn, \tag{1.3}$$

which suddenly throws a slightly different light on the issue of its convergence and possible identity with f. From a Lebesgue integral perspective, this integral only makes sense, independently of t, for integrable functions over the integers, i.e. for sequences $\hat{f} \in l^1(\mathbb{Z})$. And, in that case, this integral looks exactly analogous to the Fourier transform formula, that takes $f \in L^1(\mathbb{T})$ to its Fourier coefficients $\hat{f}(n)$,

$$\mathcal{F}(f)(n) = \hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)e^{-int}dt,$$

except for an insignificant change of sign in the exponential terms in (1.3).

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In fact, if we now observe that $(\mathbb{Z}, +)$, with the discrete topology, is also a locally compact abelian group, and that its Haar measure is exactly the counting measure, then the Fourier transform of functions in $l^1(\mathbb{Z})$ should exactly be given by some sort of integral like (1.3) over the space of integers $\mathbb{Z}$ yielding a (L^∞!) function on $\mathbb{T}$.

In other words, if the universe had worked in our favor, we would have a fully symmetrical theory where f and $\hat{f}$ would be each other's Fourier transforms, over each one's locally compact abelian group domain. Denoting by $d\tilde{t} = dt/2\pi$ the normalized Haar measure on $\mathbb{T}$ that we have been considering, we would then have

$$f \in L^1(\mathbb{T}) \to \hat{f}(n) = \int_{\mathbb{T}} f(t)e^{-int}d\tilde{t} = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)e^{-int}dt \in l^\infty(\mathbb{Z}),$$

and

$$\hat{f} \in l^1(\mathbb{Z}) \to f(t) = \int_{\mathbb{Z}} \hat{f}(n)e^{int}dn = \sum_{-\infty}^{\infty} \hat{f}(n)e^{int} \in L^\infty(\mathbb{T}).$$

This symmetry of Fourier transforms is the central idea of Pontryagin's duality theory, in abstract harmonic analysis: that, to functions on a locally compact abelian group, there corresponds a Fourier transform which maps them to functions on a dual locally compact abelian group. And that the inverse map is precisely also given by the Fourier transform on that dual group. The integers $\mathbb{Z}$ and the circle $\mathbb{T}$ are then dual groups.

Unfortunately, however, this symmetry does not hold perfectly. As we saw in the previous lesson, the best that we can expect from the Fourier coefficients of $f \in L^1(\mathbb{T})$ is that they decay to zero, from the Riemann-Lebesgue lemma. But not that they are in $l^1(\mathbb{Z})$, in order for the Fourier series to be conveniently interpreted as a properly defined Lebesgue integral over the integers $\mathbb{Z}$. In fact, $\hat{f} \in l^1(\mathbb{Z})$ corresponds to absolutely convergent series which, we also mentioned last lesson, yields continuous functions on $\mathbb{T}$ as a result of uniform convergence. And, as we already know, $C(\mathbb{T})$ and $L^1(\mathbb{T})$ sit on opposite extremes of the regularity hierarchy, with the former being a very small subset of the latter. In other words, most of the functions in $L^1(\mathbb{T})$ - even many continuous functions - will not have absolutely convergent Fourier series.

So, if we want to recover every function $f \in L^1(\mathbb{T})$ from its Fourier coefficients, we cannot hope to sum the Fourier series as an $l^1(\mathbb{Z})$ integral and we must instead devise a *summability method*: some form of assigning meaning to the sum (1.1), that coincides with the usual one when $\hat{f} \in l^1(\mathbb{Z})$, but which also extends the definition even when it does not make sense as an absolutely convergent series. Of course, the classical definition for summing series does exactly that, by taking the limit of the partial sums. It is a summability method in the sense that it coincides with the Lebesgue integral definition over the integers, but if a series is not absolutely convergent, i.e. Lebesgue integrable, and the limit (1.2) exists it nevertheless still assigns a value to the sum of the series. We call it then conditional convergence of a series and, from advanced calculus, it is well known to be highly unstable, for example from Riemann's theorem regarding changes of the order of summation.

But observing, as we did at the end of the last lesson, that the partial sums of the Fourier series correspond to the convolution of f with the Dirichlet kernel $D_N(t) = \sum_{n=-N}^{N} e^{int}$,

$$S_N[f](t) = \sum_{n=-N}^{N} \hat{f}(n)e^{int} = f * \sum_{n=-N}^{N} e^{int} = f * D_N(t), \tag{1.4}$$

this would then lead immediately to the convergence of Fourier series to f in L^p norms, had it been the case that D_N were an approximate identity. However, it is not, because the sequence of its L^1 norms, $\|D_N\|_{L^1(\mathbb{T})}$ - the so called Lebesgue constants - is unbounded, a property which will be left as an exercise. And this is the crucial point that makes partial sum convergence of Fourier series a very ineffective and difficult method for inverting the Fourier transform in order to recover the original function, pointwise or in L^p norm.

Nevertheless, from the more general and powerful vantage point that we have acquired, by studying approximate identities in Lesson 11, we can now easily guess what the solution to our problem should be. We should substitute the Dirichlet kernel by a different sequence of trigonometric polynomials that do form an approximate identity. In fact, from Proposition 1.7, at the end of Lesson 13, we know that the convolution of any trigonometric polynomial with $f \in L^1(\mathbb{T})$ will be another polynomial whose coefficients involve the terms $\hat{f}(n)$. So, if a sequence of trigonometric polynomials can be found which really is an approximate identity, then its convolution with f will depend only on the values of $\hat{f}$ and converge to f in the $L^p(\mathbb{T})$ norm. We will thus have reconstructed f from its Fourier coefficients.

Needless to say that this elegant and general way of thinking about the problem was not how it historically was first solved. At the beginning of the twentieth century, mathematicians first arrived at convolutions with approximate identities by thinking of alternative summations methods for the Fourier series, other than taking limits of the partial sums.

One way to think about general summability methods is to imagine multiplying the coefficients of a series by a sequence of terms, depending on a parameter, that force it to be absolutely convergent even when it initially is not. And then make the parameter converge to a limit at which point the multiplying factors all become equal to one, corresponding then to the full series (1.1). For example, the partial sums can be thought of as a cut-off in the frequencies between $-N$ and N applied to the Fourier series

$$\sum_{n=-N}^{N} \hat{f}(n)e^{int} = \sum_{n=-\infty}^{\infty} \chi_{[-N,N]}(n)\hat{f}(n)e^{int},$$

where $\chi_{[-N,N]}$ is the characteristic function, over the integers, of the interval $[-N, N]$. The parameter here is evidently N which, as we take the limit $N \to \infty$, makes $\chi_{[-N,N]}(n)$ converge to the constant sequence equal to one, thus fully eliminating the cut-off and yielding the complete series (1.1). So the classical way of summing series is one particular choice of summability method in this sense. Here is another one: multiply the Fourier coefficients by a fast decaying negative exponential term $e^{-\varepsilon|n|}$ for $\varepsilon > 0$,

$$\sum_{n=-\infty}^{\infty} e^{-\varepsilon|n|}\hat{f}(n)e^{int}.$$

As the sequence of Fourier coefficients is bounded, $e^{-\varepsilon|n|}\hat{f}(n) \in l^1(\mathbb{Z})$ and so these series are absolutely and uniformly convergent for every $\varepsilon > 0$ (of course they are not trigonometric polynomials, as the partial sums are, but being absolutely convergent they work equally well). In this case, we should obviously take the limit $\varepsilon \to 0$ and this, generally, yields better summing results than the limit of the partial sums. It is called Abel summability.

To understand why Abel summability is generally better than partial sum convergence, just recall from Lesson 13 that the Fourier transform of the convolution of two functions is the product of the Fourier transforms. So the multiplication of the Fourier coefficients $\hat{f}(n)$ by any summable sequence with a parameter, say $k_\varepsilon(n)$, should correspond, on the circle $\mathbb{T}$ side, to the convolution of f and a family of functions K_ε whose Fourier coefficients are $\widehat{K_\varepsilon}(n) = k_\varepsilon(n)$. We should thus have

$$S[f * K_\varepsilon] \sim \sum_{n=-\infty}^{\infty} k_\varepsilon(n)\hat{f}(n)e^{int},$$

with the final observation that, if the sequence $k_\varepsilon(n)$ on the frequency side is such that $\lim_{\varepsilon\to 0} k_\varepsilon(n) = 1$, to make the modified series converge to the full Fourier series, then on the circle side we should have

$\lim_{\varepsilon\to 0} K_\varepsilon = \delta$. In other words, summability methods, on the frequency side of the series, generally correspond to convolutions with approximate identities, on the circle side. And, of course, what makes Abel summability more effective than partial sum convergence is that, on the circle side it does correspond to the convolution of f with a true approximate identity,

$$K_\varepsilon(t) = \sum_{n=-\infty}^{\infty} e^{-\varepsilon|n|} e^{int} = \frac{1 - e^{-2\varepsilon}}{1 - 2e^{-\varepsilon}\cos t + e^{-2\varepsilon}},$$

even though this infinite series is not a polynomial, whereas the partial sum convergence corresponds to the convolution with the Dirichlet kernel

$$D_N(t) = \sum_{n=-N}^{N} e^{int} = \frac{\sin(N+\frac{1}{2})t}{\sin\frac{t}{2}}.$$

These arguments have not been rigorous up to this point, but the general idea helps to intuitively understand the connection between summability methods, convolution kernels and approximate identities[1].

Therefore, the fine line that separates a summability method from being effective, like the Abel summability, or not, like the convergence of partial sums, reduces to whether the corresponding convolution kernel is an approximate identity or not. From a very formal and nonrigorous perspective, as above, they all seem to perform the same job equally well, because their Fourier coefficients on the frequency side all converge to one as the parameter tends to the limit, and therefore all the convolution kernels should, in some way, converge to the Dirac delta on the circle $\mathbb{T}$ side. Of course, the subtlety lies in the strength of the convergence being demanded. Convergence in L^p norm is rather strong, and therefore it requires a definition of approximate identity, as we presented in Lesson 11, which the Dirichlet kernel does not fulfill. Two important observations need to be made, though.

(1) The fact that the Dirichlet kernel is not an approximate identity *does not* mean that the partial sums do not converge to f in the L^p norm. It just means that, with partial sums and the Dirichlet kernel, we cannot just simply apply the theorem of L^p convergence of approximate identities in a straightforward manner. In fact the partial sums of Fourier series do converge in the L^p norm, for $1 < p < \infty$, as we will see later in the course, as a consequence of one of the most important theorems in harmonic analysis: Riesz's theorem on the L^p boundedness of the conjugation operator, equivalent to the L^p boundedness of the Hilbert transform.

(2) If we relax the convergence requirements to weaker topologies, for example to the very weak form in the sense of distributions, then it becomes almost trivial to prove that the partial sums of Fourier series do indeed converge to f in this weak topology.

Now that the general picture of summability methods is understood, we can focus on specific examples, starting with what is arguably the best known, and historically the first such method: the Cesàro means and corresponding Fejér kernel. The Cesàro means consist of the sequence of arithmetic means of partial sums of the Fourier series

$$\sigma_N(f) = \frac{S_0[f] + S_1[f] + \cdots + S_N[f]}{N+1}. \tag{1.5}$$

It is of course an elementary fact that whenever a sequence converges, the sequence of its arithmetic means also converges to the same limit,

$$a_N \to a \Rightarrow \frac{a_0 + a_1 + \cdots + a_N}{N+1} \to a.$$

[1]Approximate identities that arise from summability methods are also called summability kernels.

But not the converse, so that the sequence of means generally converges in a broader range of cases. A trivial example is the sequence $a_N = (-1)^N$ which does not converge, but whose arithmetic means

$$\frac{a_0 + a_1 + a_2 + \cdots a_N}{N+1} = \frac{1 - 1 + 1 - 1 + 1 - \cdots + 1}{N+1} \to 0.$$

If we now use the linearity property of the convolution and (1.4), we can write the Cesàro means (1.5) as a convolution

$$\sigma_N(f) = f * \frac{D_0 + D_1 + \cdots + D_N}{N+1},$$

where the kernel

$$K_N(t) = \frac{D_0(t) + D_1(t) + \cdots + D_N(t)}{N+1} = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e^{int}, \tag{1.6}$$

is called the *Fejér kernel.* It is a sequence of trigonometric polynomials, equal to the arithmetic means of the Dirichlet kernel. We therefore have

$$\sigma_N(f)(t) = f * K_N(t) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \hat{f}(n)\, e^{int}. \tag{1.7}$$

Observe that, unlike the Dirichlet kernel, which has Fourier coefficients with an abrupt cutoff at frequencies $-N$ and N, the Fejér kernel has coefficients which correspond to a triangular cut-off, with value one at the frequency $n = 0$, down to zero, at frequencies $n = \pm N$. This slightly smoother cut-off in frequency space makes all the difference, because it turns the Fejér kernel into a true summability kernel. In fact, from (1.6) it is clear that $K_N(t) \geq 0$ for all $t \in \mathbb{T}$. Therefore, from (1.6), we have

$$\frac{1}{2\pi} \int_{\mathbb{T}} K_N(t)\, dt = \|K_N\|_{L^1(\mathbb{T})} = 1,$$

and so properties (1) and (2) in the definition of approximate identity (with the obvious adaptation from $\mathbb{R}^n$ to $\mathbb{T}$), in Lesson 11, are satisfied There remains to show property (3), for which it is helpful to have the explicit formula for the Fejér kernel

$$K_N(t) = \frac{1}{N+1} \left(\frac{\sin \frac{N+1}{2} t}{\sin \frac{t}{2}} \right)^2.$$

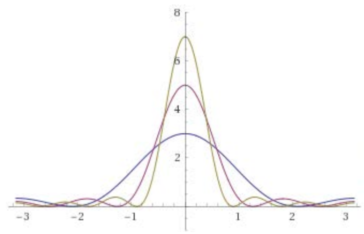
This can be easily computed from (1.6) (see Katznelson [1, 2], Chapter I, Section 2.5). And so, for any $\delta > 0$ and $|t| \geq \delta$ we have

$$K_N(t) \leq \frac{1}{N+1} \frac{1}{\left(\sin \frac{\delta}{2}\right)^2},$$

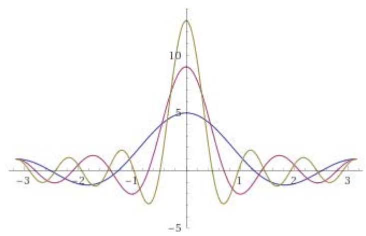
which immediately shows that

$$\int_{|t| \geq \delta} |K_N(t)|\, dt \leq \frac{2\pi}{N+1} \frac{1}{\left(\sin \frac{\delta}{2}\right)^2} \to 0,$$

as $N \to 0$, and that is property (3) in the definition of approximate identity. It is worthwhile comparing the graphs of the Fejér kernel, which are plotted for $N = 2, 4$ and 6 in the following figure



with the analogous graphs of the Dirichlet kernel



Coupling the fact that the Fejér kernel is an approximate identity, with Theorem 1.2 in Lesson 11, we can then conclude the fundamental theorem for the convergence of Cesàro means.

Theorem 1.1. *Let $f \in L^p(\mathbb{T})$, with $1 \leq p < \infty$. Then, the sequence of Cesàro means of f given by* (1.5) *and* (1.7) *converges to f in the $L^p(\mathbb{T})$ norm. If $f \in C(\mathbb{T})$ then the convergence is uniform on $\mathbb{T}$, i.e. in the $L^\infty(\mathbb{T})$ norm.*

This is the powerful theorem that we have been seeking as it yields the recovery of f from the sequence of its Fourier coefficients. Two very important corollaries immediately follow. The first one is a consequence of the fact that, if a sequence converges, than the sequence of its arithemtic means necessarily has to converge to the same limit. So that, once we now know that the Cesàro means converge in norm to f, then if the partial sums are known to converge, necessarily their limits have to be the same.

Corollary 1.2. *Let $f \in L^p(\mathbb{T})$, with $1 \leq p < \infty$. Then, if the partial sums of the Fourier series of f converge in the $L^p(\mathbb{T})$ norm, necessarily their limit has to be f. The same conclusion holds if $f \in C(\mathbb{T})$ and the convergence of the Fourier series is known to hold uniformly on $\mathbb{T}$, i.e. in the $L^\infty(\mathbb{T})$ norm.*

Observe that we *are not* stating that the Fourier series converges. This is only a conclusion about the necessary value of the limit, if the convergence is known to hold a priori. So, either the partial sums of Fourier series diverge, or, if they converge in norm, their limit can only be f.

Another major consequence of the convergence of Cesàro means in norm is the injectivity of the Fourier transform operator $\mathcal{F} : L^1(\mathbb{T}) \to l^\infty(\mathbb{Z})$.

Corollary 1.3. *Let $f \in L^1(\mathbb{T})$. If $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$ then $f = 0$. Equivalently, if $f, g \in L^1(\mathbb{T})$ are such that $\hat{f}(n) = \hat{g}(n)$ for all $n \in \mathbb{Z}$, then $f(t) = g(t)$ a.e. on $\mathbb{T}$.*

Notice, however, that one should not confuse the uniqueness of the Fourier coefficients, as stated in this corollary - that to each function in $L^1(\mathbb{T})$ there corresponds its own unique frequency fingerprint in the form of a sequence of Fourier coefficients - with the uniqueness of representation by a trigonometric series - that to each representation of a function by a trigonometric series there corresponds a unique sequence of coefficients, which should naturally be its Fourier coefficients. These are completely independent issues and, while the first one is true from the previous corollary, the second might actually be false, depending on the type of convergence being considered. To begin with, we are not at all saying that there is such a representation $f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int}$, in any traditional sense of convergence of the series (except for its Cesàro summability), from which the injectivity of the Fourier transform is being proved, let alone pointwise convergence of its partial sums. So the uniqueness of Fourier coefficients from the previous corollary is totally unrelated to the representation of functions by trigonometric series, as it holds even for functions whose Fourier series diverge at every point.

On the other hand, even if there exists a trigonometric series such that

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int},$$

for example in a pointwise sense of convergence, it is not possible to prove, in general, that $c_n = \hat{f}(n)$. Actually, trigonometric series can exhibit two quite surprising features, as we will see along this course: a trigonometric series might very well converge pointwise for all $t \in \mathbb{T}$ but to a function that is not even in $L^1(\mathbb{T})$, for which it does not make sense therefore to talk about Fourier coefficients; and different sequences of coefficients can always be found for which the corresponding trigonometric series converge almost everywhere to the same function, so that, even if that function is in $L^1(\mathbb{T})$, not only are the coefficients not unique, some of them are are not the function's Fourier coefficients either.

It is interesting to compare these phenomena with what happens for power series, where the exact opposite occurs. For power series the coefficients of the representation are unique, obtained by successively taking derivatives at the central point of the radius of convergence so that there is only one possible power series representation of a smooth function: the Taylor series. On the other hand, several different smooth (nonanalytic) functions can have the same sequence of derivatives at a point, so that the map from smooth functions to their Taylor series coefficients is not injective.

In spite of the previous observations, for trigonometric series that converge in the $L^1(\mathbb{T})$ norm its coefficients necessarily are the Fourier coefficients.

Proposition 1.4. *Let $\sum_{-\infty}^{\infty} c_n e^{int}$, with $c_n \in \mathbb{C}$, be an $L^1(\mathbb{T})$ convergent trigonometric series, in the sense that there exists $f \in L^1(\mathbb{T})$ such that the symmetric partial sums of the series converge to f in the $L^1(\mathbb{T})$ norm. Then, $\hat{f}(n) = c_n$ for all $n \in \mathbb{Z}$.*

Proof. We have

$$\left\| \sum_{j=-N}^{N} c_j e^{ijt} - f \right\|_{L^1(\mathbb{T})} \to 0,$$

as $N \to \infty$. Therefore, as this difference is a sequence that converges to 0 in $L^1(\mathbb{T})$, from the properties of Fourier coefficients seen in Lesson 13 (Corollary 1.5) we then know that its Fourier coefficients converge to 0 in $l^\infty(\mathbb{Z})$, i.e.

$$\mathcal{F}\left(\sum_{j=-N}^{N} c_j e^{ijt} - f \right)(n) \to 0,$$

uniformly in $n \in \mathbb{Z}$ as $N \to \infty$. Finally, from the linearity of the Fourier transform and the simple fact that for a trigonometric polynomial $\sum_{-N}^{N} c_n e^{int}$ the sequence of its Fourier coefficients is precisely c_n, for $-N \le n \le N$, and 0 for $|n| > N$, we conclude that

$$c_n = \mathcal{F}(f)(n) = \hat{f}(n),$$

for all $n \in \mathbb{Z}$. And this concludes the proof. □

Recall that $L^1(\mathbb{T})$ is the largest space in the hierarchy of $L^p(\mathbb{T})$ and $C^k(\mathbb{T})$ spaces, with the weakest norm, bounded above by all the others. So that, if a trigonometric series converges in any other $L^p(\mathbb{T})$ space, with $1 \le p \le \infty$, then it also converges in $L^1(\mathbb{T})$ and the same result as in the previous proposition holds. This conclusion is equally true for trigonometric series that converge in $C(\mathbb{T})$ with the supremum norm, i.e. uniformly, as well as absolutely, which is even stronger due to the Weierstrass M-test.

Another very important result that follows from the convergence of the Cesàro means is the density of trigonometric polynomials in $L^p(\mathbb{T})$ and $C(\mathbb{T})$, the latter being considered a version of the Weierstrass approximation theorem for trigonometric polynomials.

Corollary 1.5. *The set of trigonometric polynomials is dense in $L^p(\mathbb{T})$, for $1 \leq p < \infty$ and in $C(\mathbb{T})$ with the supremum ($= L^\infty$) norm.*

As the trigonometric polynomials have Fourier coefficients which are zero, for frequencies larger than their degree, the Riemann-Lebesgue Lemma is a trivial fact for them. On the other hand, by the density result above, we can approximate any function f in $L^1(\mathbb{T})$ arbitrarily closely by trigonometric polynomials. So, we could also have proved the Riemann-Lebesgue Lemma for $L^1(\mathbb{T})$ functions by approximation with polynomials, instead of approximation by differentiable functions and integration by parts, as we did in the previous lesson (see Katznelson's proof of the Riemann-Lebesgue Lemma in [1, 2]).

To finish this lesson, we will just mention a couple of other important summability kernels. The first of them is the *Poisson kernel*, given by

$$P_r(t) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{int} = \frac{1 - r^2}{1 - 2r\cos t + r^2},$$

where the series converges absolutely or any $0 < r < 1$, yielding the function written explicitly on the right hand side of this identity. For those that recall complex analysis, the Poisson kernel is used to generate harmonic functions in the interior of the unit disk of the complex plane, from prescribed values on the boundary circle. It has the disadvantage, when compared to the Fejér kernel, of not being a trigonometric polynomial, but on the other hand as an absolutely convergent trigonometric series it remains equally convenient for computations. And it is not hard to prove that it is a summability kernel, i.e. an approximate identity, as $r \to 1$. Therefore we obtain analogous convergence results as in Theorem 1.1 for the convolution

$$f * P_r(t) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) e^{int},$$

which are called the *Abel means* of the Fourier series, in analogy with the Cesàro means. In fact, if one makes $r = e^{-\varepsilon}$ this is precisely the Abel summability method that we saw before. In other words, the Poisson kernel is the approximate identity kernel corresponding to the Abel summability method. The Poisson kernel is going to be a central ingredient in the important proof, later in the course, of Riesz's theorem on the L^p convergence of the partial sums of Fourier series, by using the complex analysis methods related to harmonic functions on the unit disk and the conjugation operator.

The second approximate identity kernel worth mentioning is the *de la Valée Poussin kernel*

$$V_N(t) = 2K_{2N+1}(t) - K_N(t).$$

Just like the Fejér kernel, the de la Valée Poussin kernel is also an approximate identity based on trigonometric polynomials, in this case of degree $2N + 1$. Their advantage stems from the fact that, on the frequency side, they correspond to the difference of two Fejér triangular cut-offs, one twice as large and wide as the other, so that the resulting effect is that we obtain a plateau of Fourier coefficients equal to one, between the frequencies $\pm(N + 1)$,

$$\widehat{V_N}(n) = 1 \quad \text{for} \quad -N - 1 \leq n \leq N + 1,$$

and this is often a useful property because its convolution with f will then keep the Fourier frequencies of f unchanged on this plateau.

References

[1] Yitzhak Katznelson *An Introduction to Harmonic Analysis,* 2nd Edition, Dover Publications, 1976.
[2] Yitzhak Katznelson *An Introduction to Harmonic Analysis,* 3rd Edition, Cambridge University Press, 2004.